# Total Differential Equations

An equation of the form $Pdx + Qdy + Rdz = 0$ in which $P, Q, R$ are functions of $x, y$ and $z$ is called a total differential equation or single differential equation or pfaffian differential equation.

If we are a given a relation of the form

$f(x, y, z) = c$ — ①

where $c$ is arbitrary constant, then we can write

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = 0 \quad \text{— ②}$$

If the quantities $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ have a common factor, then the relation ② can be simplified by cancelling that factor throughout and ② takes the form

$$P dx + Q dy + R dz = 0 \quad \text{— ③}$$

in which $P, Q, R$ are functions of $x, y, z$.

Thus, if a relation ① be given, then from it, we can find a relation of the type ③, which is a total differential equation.

Th The necessary and sufficient condition for the integrability of a pfaffian Diff$^{l}$ equation $Pdx + Qdy + R dz = 0$ is

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$

The condition is necessary :-

Let us consider the pfaffian differential equ

$P\,dx + Q\,dy + R\,dz = 0$ — ①

Where $P, Q, R$ are functions of $x, y, z$

To deduce the necessary condition we assume that ① is integrable and let

$\phi(x,y,z) = c$ — ②

be its integral where $c$ is an arbitrary constant. Then the total diff$^{l}$ $d\phi$ must be equal to $(P\,dx + Q\,dy + R\,dz)$ or $\mu(P\,dx + Q\,dy + R\,dz)$ where $\mu$ is called an integrating factor, is a function of $x, y, z$.

But $d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$ :

Thus equating with $d\phi = \mu(P\,dx + Q\,dy + R\,dz)$

we get $\frac{\partial \phi}{\partial x} = \mu P$ ; $\frac{\partial \phi}{\partial y} = \mu Q$ ; $\frac{\partial \phi}{\partial z} = \mu R$.
③ — ④ — ⑤

Let $\phi(x,y,z)$ be such that

$$\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial y}\right) = \frac{\partial^2 \phi}{\partial z \partial y} = \frac{\partial^2 \phi}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial z}\right) \text{ etc.}$$

now using ④ and ⑤

$$\frac{\partial}{\partial z}(\mu Q) = \frac{\partial}{\partial y}(\mu R)$$

$$\Rightarrow \frac{\partial \mu}{\partial z} Q + \mu \frac{\partial Q}{\partial z} = \mu \frac{\partial R}{\partial y} + R \frac{\partial \mu}{\partial y}$$

$$\Rightarrow \mu\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial R}\right) = R\frac{\partial \mu}{\partial y} - Q\frac{\partial \mu}{\partial z} \quad \text{— ⑥}$$

Similarly considering ⑤, ③ and ③, ④ we

get two other relations as

$$\mu\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) = P\frac{\partial \mu}{\partial z} - R\frac{\partial \mu}{\partial x} \quad -(7)$$

and $$\mu\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial z}\right) = Q\frac{\partial \mu}{\partial x} - P\frac{\partial \mu}{\partial y} \quad -(8)$$

multiplying (6), (7) and (8) by P, Q, R resp.

and adding we get

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0 \quad -(9)$$

This is the necessary condition to be satisfied by P, Q, R so that the equation (1) will have an integral of the form

$\phi(x, y, z) = $ constant.

** Geometrical interpretation of $Pdx + Qdy + Rdz = 0$

This differential equation shows that two straight lines whose direction cosines are proportional to dx, dy, dz and P, Q, R are perpendicular to each other. Now the direction cosines of the tangent to a curve at a point (x, y, z) are proportional to dx, dy, dz. Hence the above equation expresses that the ~~tang~~ tangent to a curve at the point (x, y, z) is perpendicular to a straight line whose d.c.s are proportional to P, Q, R.

## Problems

(1) Examine if the equation $(y+z)dx + dy + dz = 0$ is integrable.

Soln:- Comparing the given equation with $Pdx + Qdy + Rdz = 0$ we get

$$P = (y+z) \,;\, Q = R = 1$$

now $P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

$$= (y+z)\cdot(0) + 1(-1) + 1\cdot 1 = 0.$$

Thus the condition of integrability is satisfied so, given equation is integrable.

(2) Examine if the equation $(y^2+yz)dx + (z^2+zx)dy + (y^2-xy)dz = 0$ is integrable or not.

Soln:- Here $P = (y^2+yz)$ ; $Q = (z^2+zx)$ and

$$R = (y^2-xy)$$

now, $P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

$$= (y^2+yz)\left[\frac{\partial}{\partial z}(z^2+zx) - \frac{\partial}{\partial y}(y^2-xy)\right]$$
$$+ (z^2+zx)\left[\frac{\partial}{\partial x}(y^2-xy) - \frac{\partial}{\partial z}(y^2+yz)\right]$$
$$+ (y^2-xy)\left[\frac{\partial}{\partial y}(y^2+yz) - \frac{\partial}{\partial x}(z^2+zx)\right]$$

$$= (y^2+yz)(2z + x - 2y + x) + (z^2+zx)(-y - y)$$
$$+ (y^2-xy)(2y + \cancel{z} - \cancel{z})$$

$$= (y^2 + yz)(2x - 2y + 2z) - 2y(z^2 + zx) + 2y(y^2 - xy)$$

$$= 2(y^2 + yz)(x - y + z) - 2(yz^2 + xyz) + 2(y^3 - xy^2)$$

$$= 2\left[\cancel{xy^2} - \cancel{y^3} + \cancel{zy^2} + \cancel{xyz} - \cancel{y^2z} + \cancel{yz^2} - \cancel{yz^2} - \cancel{xyz} + \cancel{y^3} - \cancel{xy^2}\right]$$

$$= 0$$

Thus the condition of integrability is satisfied so, given equation is integrable.

Test the condition of integrability of the given equations —

① $\dfrac{yz}{x^2+y^2}\,dx - \dfrac{xz}{x^2+y^2}\,dy - \tan^{-1}\left(\dfrac{y}{x}\right)dz = 0$

② $(2xz - yz)\,dx + (2yz - xz)\,dy - (x^2 - xy + y^2)\,dz = 0$

③ $y^2z(y+z)\,dx + zx(z+x)\,dy + x^2y(x+y)\,dz = 0.$

④ $z^2(z-y)\,dx + z(z+x)\,dy + x(x-y)\,dz = 0$

⑤ $(y^2 + z^2 - x^2)\,dx - 2xy\,dy - 2xz\,dz = 0$

⑥ $2yz\,dx - 2xz\,dy - (x^2 - y^2)(z-1)\,dz = 0$

⑦ $(1 + yz)\,dx + x(z - x)\,dy - (1 + xy)\,dz = 0.$

Ans → ③ and ④ are not integrable.

①, ②, ⑤, ⑥, ⑦ are integrable.